Stability Analysis of Nonlinear Differential Autonomous Systems with Applications to Flutter

Linda L. Smith*
Massachusetts Institute of Technology, Cambridge, Mass.
and
Luigi Morino†
Boston University, Boston, Mass.

This paper deals with the general theory of stability analysis of nonlinear differential autonomous systems of the type $\dot{x}=A(\lambda)x+f(x, \lambda)$, with the linear system stable (unstable) for $\lambda<\lambda_0$ ($\lambda>\lambda_0$). The analysis is performed by a singular perturbation method, the multiple time scaling. The solution is obtained in the form of an asymptotic expansion in the neighborhood of the stability boundary, $\lambda=\lambda_0$. It is shown that there always exists a limit cycle, either for $\lambda<\lambda_0$ (unstable limit cycle) or for $\lambda>\lambda_0$ (stable limit cycle). The analytical expression for the limit cycle amplitude also is given. Applications to nonlinear flutter of panels and wings are included. Numerical results are in excellent agreement with existing ones.

Nomenclature

a	= transient function, Eq. (22)
$\lvert a \rvert_{LC} = \lvert \beta_R/\gamma_R \rvert^{1/2}$	= limit cycle value of $\lvert a \rvert$
$A(\lambda)$	= matrix of linear terms, Eq. (1)
A_n	= coefficients of expansion of A, Eq. (6)
b_{npq}	= coefficients of second-order nonlinear terms, Eq. (2)
c_{npqr}	= coefficients of third-order nonlinear terms, Eq. (2)
$f(x,\lambda)$	= vector of nonlinear terms, Eq. (1)
k	= initial-condition constant, Eq. (23)
u	= right eigenvector of A, Eq. (12)
v	= left eigenvector of A, Eq. (18)
x	= vector of unknowns, Eq. (1)
x_n	= coefficient of expansion of x, Eq. (3)
z_3	= see Eq. (9)
β	= see Eq. (20)
γ	= see Eq. (21)
λ	= stability parameter
λ_n	= coefficients of expansion of λ, Eq. (5)
τ	= time
$\tau_n = \epsilon^n \tau$	= multiple time scales
τ_{20}	= $-(\ln \lvert k \rvert)/(2\beta_R)$
ω	= limit cycle frequency, Eq. (34)
ω_n	= coefficients of expansion of ω, Eq. (34)

Introduction

THIS paper deals with the general theory of stability analysis of nonlinear differential autonomous systems. The analysis is performed by a singular perturbation method, the multiple time scaling. The solution is obtained in the form of an asymptotic expansion. Applications to several nonlinear flutter problems are included.

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Index categories: Structural Stability Analysis; Aeroelasticity and Hydroelasticity.

*Research Assistant. Student Member AIAA.

†Director, Computational Continuum-Mechanics Program, College of Engineering. Member AIAA.

In order to discuss the analysis presented here, it is convenient to introduce the concepts of stable and unstable limit cycles. This can be accomplished best by making reference to the well-known difference between benign and explosive flutter. In benign flutter, the system, beyond its stability boundary, starts to oscillate, but the oscillations reach a finite amplitude (stable limit cycle); failure would then be due to fatigue. In explosive flutter, the system, beyond its stability boundary, starts to oscillate, and the oscillation grows until failure occurs; as a matter of fact, this unlimited growth might occur even below the stability boundary, if an initial "kick" of sufficient amplitude is experienced by the system (unstable limit cycle). This paper is an extension of a method for analyzing nonlinear panel flutter,[1] which can predict qualitatively and quantitatively the limit-cycle behavior described previously. Further developments of the method were presented in Refs. 2-5. The limitation, in all of these works, is that the mathematical model of the system is a set of nonlinear differential equations of second order in time (Lagrange's equation of motion, for instance). This makes the method not sufficiently flexible to analyze equations of different order. Furthermore, even for second-order systems, fully automated computer analysis is complicated if the nonlinear terms depend upon the first derivatives of the unknowns.[5]

These disadvantages are eliminated in Ref. 6 by rewriting the equations in state-variable form. In other words, the system is assumed to be described by a first-order differential system of the type

$$dx/d\tau = A(\lambda)x + f(x,\lambda) \tag{1}$$

where x is the vector of the state variables, A is the matrix of the linear terms, λ is a parameter such that for $\lambda<\lambda_0$ ($\lambda>\lambda_0$) the linear system is stable (unstable), and $f(x, \lambda)$ is the vector of the nonlinear terms of the type

$$f = \{f_n\} = \{ \sum_{pq} b_{npq}(\lambda)x_p x_q + \sum_{pqr} c_{npqr}(\lambda)x_p x_q x_r + \ldots \} \tag{2}$$

It may be noted that this analysis is an extension of the preceding one,[1] since the systems considered in Ref. 1-5 may be reduced easily in the form given by Eq. (1). The state-variable formulation is used in Ref. 7 to study flutter of panels and airfoils in supersonic flows.

Here the solution obtained in Ref. 6 is outlined briefly first. Applications to nonlinear flutter of panels and airfoils[7] then are presented. Additional results are given in Ref. 7. These results indicate that, although the method of solution is quite involved, the solution itself is very simple and easy to compute once the solution of the linear system has been obtained.

Solution of Problem

The solution of Eq. (1) is obtained in Ref. 6 up to the fifth-order nonlinear terms. For simplicity, only the third-order analysis is described here, and the second-order nonlinear terms are assumed to be equal to zero, $b_{npq}=0$. (The analysis of second-order nonlinear terms is given briefly in Appendix A.)

The analysis is performed in the neighborhood of the stability boundary, $\lambda=\lambda_0$, for which one pair of the eigenvalues of A is purely imaginary, $\pm i\omega_0$, whereas the others have negative real parts. In this case, using the multiple-time scale method[8] and following the same rationale as used in Ref. 1 (i.e., that the nonlinear terms are of the same order of magnitude as the change in the linear terms due to $\lambda-\lambda_0$), set

$$x=\epsilon x_1+\epsilon^3 x_3+O(\epsilon^5) \tag{3}$$

and introduce the multiple scales, $\tau_0=\tau$, $\tau_2=\epsilon^2\tau,\ldots$, so that

$$\mathrm{d}/\mathrm{d}\tau=\partial/\partial\tau_0+\epsilon^2(\partial/\partial\tau_2)+O(\epsilon^4) \tag{4}$$

Furthermore, letting

$$\lambda=\lambda_0+\epsilon^2\lambda_2+O(\epsilon^4) \quad (\lambda_2\pm 1) \tag{5}$$

and assuming that A depends analytically upon λ, one obtains

$$A=A_0+\lambda_2\epsilon^2 A_2+O(\epsilon^4) \tag{6}$$

where $A_0=A(\lambda_0)$, and $A_2=[\partial A/\partial\lambda]_{\lambda=\lambda_0}$. Combining Eqs. (1, 3, 4, and 6) and separating terms of same order of magnitude, one obtains

$$\partial x_1/\partial\tau_0=A_0 x_1 \tag{7}$$

$$\partial x_3/\partial\tau_0=A_0 x_3+z_3 \tag{8}$$

where

$$z_3=-\partial x_1/\partial\tau_2+\lambda_2 A_2 x_1+f_3 \tag{9}$$

with

$$f_3=\left\{\sum_{pqr} c_{npqr}(\lambda_0)x_{1,p}x_{1,q}x_{1,r}\right\} \tag{10}$$

Disregarding the damped terms[1] (corresponding to negative-real-part eigenvalues), the solution of Eq. (7) is

$$x_1=a(\tau_{2,\ldots})u e^{i\omega_0\tau_0}+\bar a(\tau_{2,\ldots})\bar u e^{-i\omega_0\tau_0} \tag{11}$$

where the bar indicates the conjugate value, and $i\omega_0$ and u are the imaginary eigenvalue and the corresponding eigenvector of A_0, i.e., the solution of

$$[i\omega_0 I-A_0]u=0 \tag{12}$$

Combining Eqs. (8) and (11), one obtains

$$\partial x_3/\partial\tau_0-A_0 x_3=z_3^{(3)}e^{i3\omega_0\tau_0}$$
$$+z_3^{(1)}e^{i\omega_0\tau_0}+\bar z_3^{(1)}e^{-i\omega_0\tau_0}+\bar z_3^{(3)}e^{-i3\omega_0\tau_0} \tag{13}$$

where

$$z_3^{(3)}=a^3 f_3^{(3)} \tag{14}$$

$$z_3^{(1)}=-(\partial a/\partial\tau_2)u+a\lambda_2 A_2 u+a^2\bar a f_3^{(1)} \tag{15}$$

with

$$f_3^{(3)}=\left\{\sum_{pqr} c_{npqr}u_p u_q u_r\right\} \tag{16}$$

$$f_3^{(1)}=\left\{\sum_{pqr} c_{npqr}(u_p u_q \bar u_r+u_p\bar u_q u_r+\bar u_p u_q u_r)\right\} \tag{17}$$

This equation yields secular terms (i.e., terms of the type $\tau_0 e^{i\omega_0\tau_0}$) unless a certain condition is satisfied, i.e. (see for instance Refs. 1, 4, and 6), that $z_3^{(1)}$ be orthogonal to the left eigenvector v of the matrix A_0:

$$v^T[i\omega_0 I-A_0]=0 \tag{18}$$

This condition yields

$$\partial a/\partial\tau_2+\beta a+\gamma a^2\bar a=0 \tag{19}$$

with

$$\beta=-\lambda_2 v^T A_2 u/v^T u=\beta_R+i\beta_I \tag{20}$$

$$\gamma=-v^T f_3^{(1)}/v^T u=\gamma_R+i\gamma_I \tag{21}$$

The solution of Eq. (19) is

$$a=|a|e^{i\varphi} \tag{22}$$

with

$$|a|=\left[\frac{-\beta_R/\gamma_R}{1+ke^{2\beta_R\tau_2}}\right]^{1/2} \tag{23}$$

and

$$\varphi=(\gamma_I\beta_R/\gamma_R-\beta_I)\tau_2+(\gamma_I/\gamma_R)\ln|a|+\varphi_0 \tag{24}$$

where k and φ_0 are determined by the initial conditions.‡

Discussion of Solution

The function $|a|/|\beta_R/\gamma_R|^{1/2}$ is plotted in Figs. 1-4 for all of the possible combinations of the signs of β_R and γ_R. Note that

$$\beta_R \lesseqgtr 0 \text{ for } \lambda_2=\pm 1 \qquad (\lambda\gtreqless\lambda_0) \tag{25}$$

since the linear system is unstable (i.e., $\beta_R<0$) for $\lambda_2=1$ (i.e., $\lambda>\lambda_0$). Note that only real values of $|a|$ are plotted.

Consider first Fig. 1, which corresponds to $\gamma_R>0$ (stabilizing nonlinear terms) and $\lambda>\lambda_0$ (i.e., $\beta_R<0$, destabilizing linear terms. In this case, there exist two branches, depending upon the sign of k. If the initial conditions are such that at time $\tau_2=0$, $|a|>|\beta_R/\gamma_R|^{1/2}$ ($|a|<|\beta_R/\gamma_R|^{1/2}$), then $k<0$ ($k>0$) and the amplitude decreases (increases) to the limit-cycle value

$$\lim_{\tau_2\to\infty}|a|=|a|_{LC}=|\beta_R/\gamma_R|^{1/2} \tag{26}$$

In other words, for $\gamma_R>0$ and $\lambda>\lambda_0$ (i.e., $\beta_R<0$), there exists a stable limit cycle; note that the solution $|a|=0$ is unstable. This behavior could have been predicted by noting that for large values of $|a|$ the stabilizing nonlinear terms are dominant, whereas for small values of $|a|$ the destabilizing linear terms are dominant [see Eq. (19)].

It is worth noting that the absolute value of k influences only the value of

$$\tau_{20}=-(\ln|k|)/(2\beta_R) \tag{27}$$

‡In the fifth-order analysis, k and φ_0 are functions of τ_4.[6]

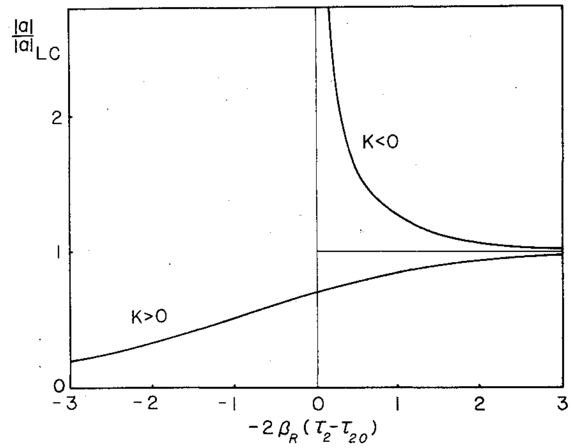

Fig. 1 **Transient response function for $\beta_R < 0$, $\gamma_R > 0$.**

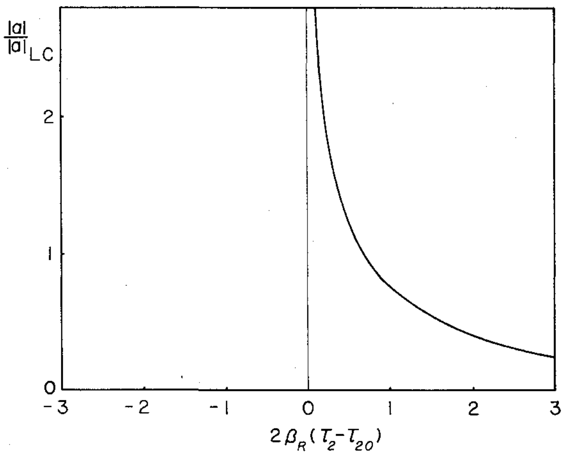

Fig. 2 **Transient response function for $\beta_R > 0$, $\gamma_R > 0$.**

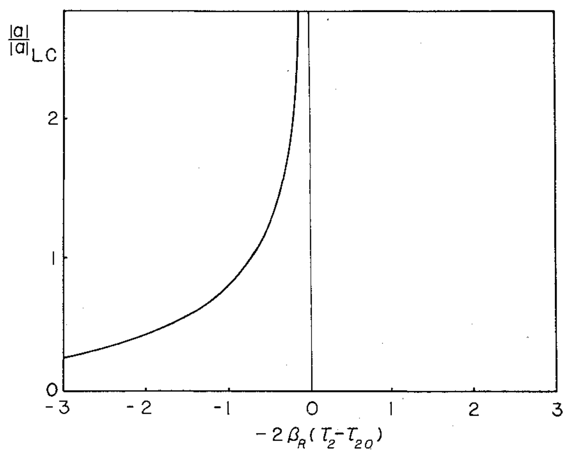

Fig. 3 **Transient response function for $\beta_R < 0$, $\gamma_R < 0$.**

which in turn determines the value of the abscissa that corresponds to the time origin $\tau_2 = 0$. (This is true for Figs. 2-4 as well).

Next consider Fig. 2, which corresponds to $\gamma_R > 0$ (stabilizing nonlinear terms) and $\lambda < \lambda_0$ (i.e.) $\beta_k > 0$, stabilizing linear terms. In this case, real values for $|a|$ are obtained only if $k < 0$ and $\tau_2 > \tau_{20}$. The solution tends to zero:

$$\lim_{\tau_2 \to \infty} |a| = 0 \tag{28}$$

i.e., the solution $|a| = 0$ is stable. This behavior is to be expected, since both linear and nonlinear terms are stabilizing.

Summarizing, if $\gamma_R > 0$ (stabilizing nonlinear terms), there exists a stable limit cycle for $\lambda > \lambda_0$ (when the linear analysis

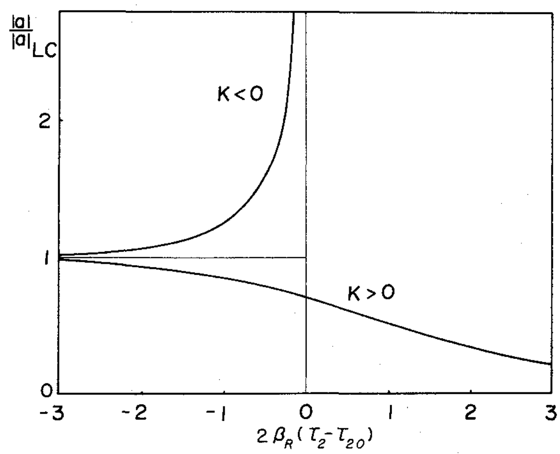

Fig. 4 **Transient response function for $\beta_R > 0$, $\gamma_R < 0$.**

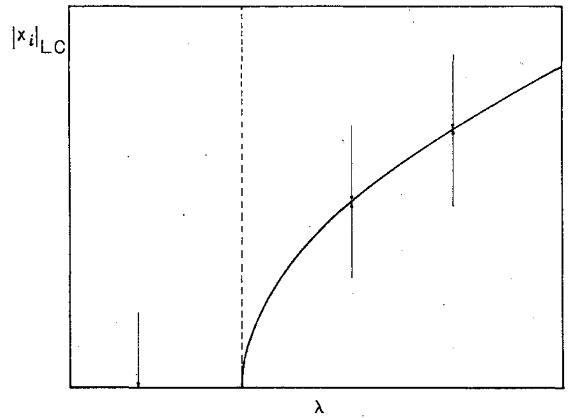

Fig. 5 **Limit cycle amplitude as a function of λ for $\gamma_R > 0$.**

predicts instability), whereas for $\lambda < \lambda_0$ the solution $|a| = 0$ is unconditionally stable.

Next consider Fig. 3, which corresponds to $\gamma_R < 0$ (destabilizing nonlinear terms) and $\lambda > \lambda_0$ (i.e., $\beta_R < 0$, destabilizing linear terms). In this case, $|a|$ is real only for $k < 0$ and $\tau_2 < \tau_{20}$. The solution tends to infinity as τ_2 tends to τ_{20}:

$$\lim_{\tau_2 \to \tau_{20}} |a| = \infty \tag{29}$$

i.e., the solution is unconditionally unstable. This behavior is expectable, since both linear and nonlinear terms are destabilizing.

Last consider Fig. 4, which corresponds to $\gamma_R < 0$ (destabilizing nonlinear terms) and $\lambda < \lambda_0$ (i.e., $\beta_R > 0$, stabilizing linear terms). Note that two branches exist, for $k \gtrless 0$, respectively. In both cases

$$\lim_{\tau_2 \to -\infty} |a| = |a|_{LC} = |\beta_R / \gamma_R|^{1/2} \tag{30}$$

However, if the initial conditions are such that $|a| > |a_{LC}|$ at time $\tau_2 = 0$ (i.e., $k < 0$), then the solution grows to infinity:

$$\lim_{\tau_2 \to \tau_{20}} |a| = \infty \quad \text{if } |a|_0 > |a|_{LC} \tag{31}$$

On the other hand, if the initial conditions are such that $|a| < |a|_{LC}$ at time $\tau_2 = 0$ (i.e., $k > 0$), then the solution goes to zero:

$$\lim_{\tau_2 \to \infty} |a| = 0 \quad \text{if } |a|_0 < |a|_{LC} \tag{32}$$

The preceding behavior may be characterized by saying that there exists an unstable limit cycle. The solution $|a| = 0$ is stable, whereas the solution $|a| = |a|_{LC}$ is unstable. In other words, the system is conditionally stable (i.e., is stable only if the initial conditions are sufficiently small, whereas it is unstable if the initial conditions are sufficiently large). This behavior could have been predicted by noting that the stabilizing linear terms dominate for small values of $|a|$ whereas distablilzing nonlinear terms dominate for large values of $|a|$. It should be noted that this behavior is very significant in practical applications, since, if the initial conditions are sufficiently large, instability occurs even if the linear analysis predicts stability.

Summarizing, if $\gamma_R<0$ (destabilizing nonlinear terms), there exists an unstable limit cycle for $\lambda<\lambda_0$ (when the linear analysis predicts stability), whereas for $\lambda>\lambda_0$ the solution is unconditionally unstable.

Note that, combining Eqs. (3, 11, 22 and 24) one obtains

$$x=\epsilon 2 \text{ Real } \{ |a| u \exp] [\omega\tau + (\gamma_I/\gamma_R)\ln|a| + \varphi_0 i\} + O(\epsilon^3) \quad (33)$$

where, according to Eq. (5), $\epsilon = |\lambda-\lambda_0|^{1/2} + O(\epsilon^3)$, and

$$\omega=\omega_0+\epsilon^2\omega_2+O(\epsilon^4) \quad (34)$$

with $\omega_2=\beta_R\gamma_I/\gamma_R-\beta_I$.

Therefore, the comments just presented may be restated as follows: there always exists a limit cycle given by

$$x=2|(\lambda-\lambda_0)\beta_R/\gamma_R|^{1/2}\text{Real}[e^{i(\omega\tau+\varphi_*)}u]+O(\epsilon^3) \quad (35)$$

where $\varphi_* = \varphi_0 + (\gamma_I/\gamma_R)\ \ln\ |\beta_R/\gamma_R|^{1/2}$ is a constant. If $\gamma_R>0$ ($\gamma_R<0$), the limit cycle exists for $\lambda>\lambda_0$ ($\lambda<\lambda_0$) and is stable (unstable). Furthermore, if $\gamma_R>0$ ($\gamma_R>0$), the system is stable (unstable) for $\lambda<\lambda_0$ ($\lambda>\lambda_0$). This behavior is indicated in Fig. 5 (Fig. 6), where the arrows indicate the variation with time.

Numerical Results

Numerical results for nonlinear flutter of panels and airfoils in supersonic flow are presented in this section.§ These problems were chosen because existing results are available in order to assess the present method. Additional results are given in Ref. 7.

Consider the panel flutter first. The governing equations, presented in Appendix B, are obtained from the ones of Dowell[9] by adding the effects of structural and external damping[1] and by using the piston theory[10] for the aerodynamic pressure. All of the numerical results presented here are obtained for Poisson's ratio $\nu_P=0.3$ and $R_x=R_y=0$ [Eq. (B6)]. Several values for the damping parameters θ_1 and θ_2 [Eq. (B5)] have been used. These values correspond to $\theta_1/\theta_2=0$, 1 and ∞, and $\theta_1+\theta_2=0.01$, 0.1, and 1.¶ The reason for doing this is that the results of Ref. 1 indicate that the solution is highly sensitive to the ratio of the damping parameters but not to their absolute values. The results are given as continuous functions of the panel aspect ratio ℓ_x/ℓ_y. The flutter value for the dynamic pressure parameter λ_0 is plotted as a function of ℓ_x/ℓ_y in Fig. 7, whereas the value of the limit-cycle amplitude $|a|_{LC}$ is plotted as a function of ℓ_x/ℓ_y in Fig. 8. These results were obtained by using six modes [$N=6$ in Eq. (B1)]. (Results for four modes coincide, within plotting accuracy, with the six-mode case: this indicates that six modes are sufficient for convergence). The two-mode results are in excellent agreement with the analytical solution obtained in Ref.1 in the limit as $(\theta_1+\theta_2)$ tends to zero. The amplitude of oscillation, $(w/h)_p$, at $y/\ell_y=1/2$ and $x/\ell_x=3/4$, is given by

$$\left[\frac{w}{h}\right]_p=2(\lambda-\lambda_0)^{1/2}|\beta_R/\gamma_R|^{1/2}\left|\sum_1^6 u_n\sin\left[\frac{n\pi 3}{4}\right]\right|$$

and is presented as a function of λ in Fig. 9 for $0.01\le\theta_1\le 5$,** $\theta_2=0$, and $\ell_x/\ell_y=0$, 1, and 2. The results are in excellent agreement with the ones obtained by Dowell.[9] It should be noted that (see Appendix B) θ_1 is treated here as a constant, whereas, for the results of Ref. 9, θ_1 is given by

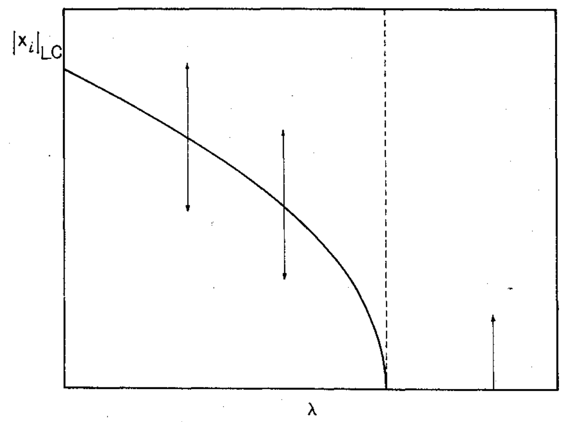

Fig. 6 Limit cycle amplitude as a function of λ for $\gamma_R<0$.

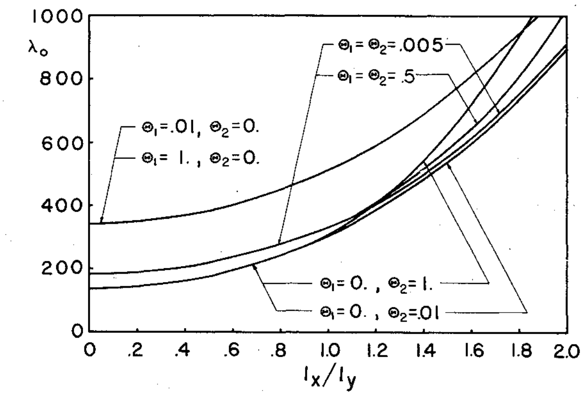

Fig. 7 Critical stability parameter λ_0, for panel flutter equations, as a function of ℓ_x/ℓ_y.

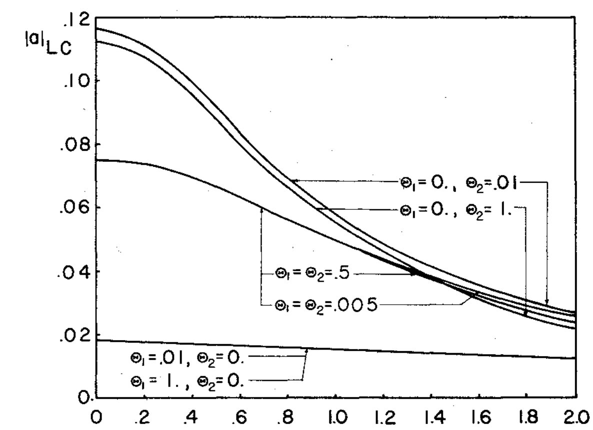

Fig. 8 Limit cycle value of $|a|$, for panel flutter equations, as a function of ℓ_x/ℓ_y.

§Note that in all the results presented here the eigenvector u is normalized by choosing $u_1=1$.

¶Results for $\theta_1+\theta_2=0.1$ coincide with the ones for $\theta_1+\theta_2=0.01$ and therefore are not shown in Figs. 7 and 8.

**The results coincide, within plotting accuracy, in this range.

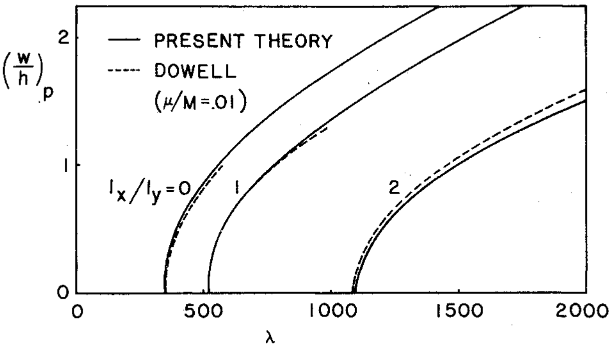

Fig. 9 Limit cycle amplitude as a function of λ_0 for panel flutter equations.

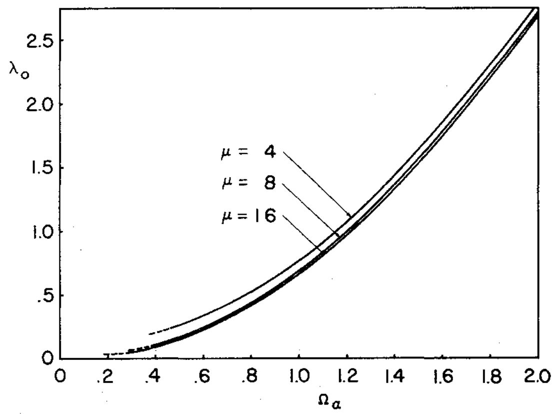

Fig. 10 Critical stability parameter λ_0, for airfoil flutter equations, as a function of Ω_α.

$\theta_l = (\lambda\mu/M)^{1/2}$ with $\mu/M = 0.01$, i.e., $\theta_l < 5$. It may be noted that the difference for higher values of $|a|$ is due to the effect of the third-order nonlinear terms, which are neglected in the solution. However, these terms are determined by the fifth-order nonlinearities in the equation,[6] and therefore their inclusion would be inconsistent with the approximations introduced in the differential equations in which fifth-order nonlinearities have been neglected.[1]

Next consider the flutter of a rigid airfoil. The governing equations are derived in Ref. 5 and presented briefly here in Appendix C. All of the numerical results presented here are obtained for $\xi_c/b = 0.2$ [Eq. (C1)], $\xi_E = 0.0$ [Eq. (C4)], $\nu = 0.21$ [Eq. (C8)], and $\Omega_\xi = 0$ [Eq. (C10)]. These values are used in Ref. 10 for the linear analysis. The results are presented for $\mu = 4$, 8, 12, and 16 [Eq. (C 9a)] and for a continuous variation of Ω_α [Eq. (C10)]. The values of λ [Eq. (C9b)] are presented in Fig. 10. From these, the Mach number M is obtained (see Fig. 11), since $\lambda = M/\mu$ [Eq. (C9b)]. These results coincide with those of Ref. 10 [see Eq. (C16)]. Note that all of the curves are truncated for those values of Ω_α corresponding to $M < 1$. The values of $|a|_{LC}$ are given in Fig. 12 for values of $\kappa_\alpha/\Omega_\alpha^2 = 0$, ± 1, and ± 2 and $\chi = 0.2$. [Eq. (C12)]. It may be noted that the aerodynamic nonlinearities ($\kappa_\alpha = 0$) are always destabilizing, whereas the structural nonlinearities are stabilizing for $\kappa_\alpha > 0$ (hard spring) and distablilzing for $\kappa_\alpha < 0$ (soft spring). Note also that, as Ω_α increases, the Mach number reaches very high values, and therefore the aerodynamic nonlinearities become much stronger than the structural nonlinearities, and thus all of the limit cycles become unstable. Also, results obtained in Ref. 5 by disregarding the nonlinear terms in the piston theory [*i.e.*, $\chi = 0$ in Eq. (C12)] indicate that these terms are small with respect to the geometric aerodynamic nonlinearities for $M \simeq 1$ but become very important for $M >> 1$.

It should be noted that the results presented here coincide with the ones presented in Ref. 5, where the solution is ob-

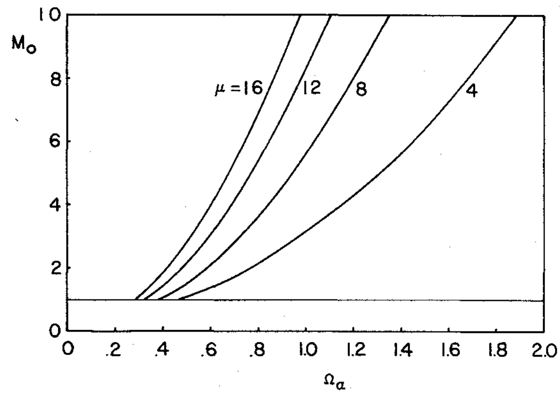

Fig. 11 Critical Mach number M_0, for airfoil flutter equations, as a function of Ω_α.

tained directly from the second-order system, Eq. (C7). The same is true for the panel flutter results of Ref. 1. This point is analyzed in detail in Appendix D, where it is shown that the solution of Ref. 1 (used also in Ref. 5) descends from the solution given here if A is of the form given by Eq. (A14).††

Conclusions

A very general formulation for the stability analysis of nonlinear autonomous systems has been presented. The analytical results indicate that the phenomenon mentioned in the Introduction (benign vs explosive flutter) is characteristic of any system described by Eq. (1), in the neighborhood of the stability boundary, $\lambda = \lambda_0$.‡‡ More precisely, if $\gamma_R > 0$, the nonlinear terms are stabilizing (Fig. 5), and the system is stable for $\lambda < \lambda_0$, whereas for $\lambda > \lambda_0$ it tends to a finite amplitude (limit cycle) proportional to $|\lambda - \lambda_0|^{1/2}$. On the other hand, if $\gamma_R < 0$, the nonlinear terms are distabilizing, (Fig. 6), and the system is unstable for $\lambda > \lambda_0$, whereas for $\lambda < \lambda_0$ the system is stable if the initial amplitude is small, but it diverges (even if the linear analysis predicts stability) if the initial amplitude is sufficiently large. In other words, the panel-flutter results have been extended here to any system described by Eq. (1). In addition, the numerical results fmc nonlinear flutter of panels and airfoils are in excellent agreement with the existing ones. Additional applications are now under consideration.

Appendix A

In the main body of this paper, Eq. (1) is analyzed under the assumption that the even-order nonlinear terms are equal to zero. The third-order solution of Eq. (1) is considered in this Appendix without the assumption $b_{npq} = 0$. The solution is given by

$$x = \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + O(\epsilon^4) \tag{A1}$$

where $x_1, x_2, x_3, \ldots$, are functions of τ_0, τ_1, τ_2. Hence,

$$\frac{d}{d\tau} = \frac{\partial}{\partial\tau_0} + \epsilon\frac{\partial}{\partial\tau_1} + \epsilon^2\frac{\partial}{\partial\tau_2} + O(\epsilon^3) \tag{A2}$$

Combining Eqs. (1, 2, 6, A1, and A2) and separating terms of the same order of magnitude, one obtains

$$\partial x_1/\partial\tau_0 - A_0 x_1 = 0 \tag{A3}$$

$$\partial x_2/\partial\tau_0 - A_0 x_2 = f_2 \tag{A4}$$

$$\partial x_3/\partial\tau_0 - A_0 x_3 = f_3 - \partial x_1/\partial\tau_2 - \partial x_2/\partial\tau_1 + \lambda_2 A_2 x_1 \tag{A5}$$

††Comparison with the harmonic balance method[4] is given in Appendix E.

‡‡Double crossing of the imaginary axis at $\lambda = \lambda_0$ is not considered. Crossing at the origin is considered in Appendix F.

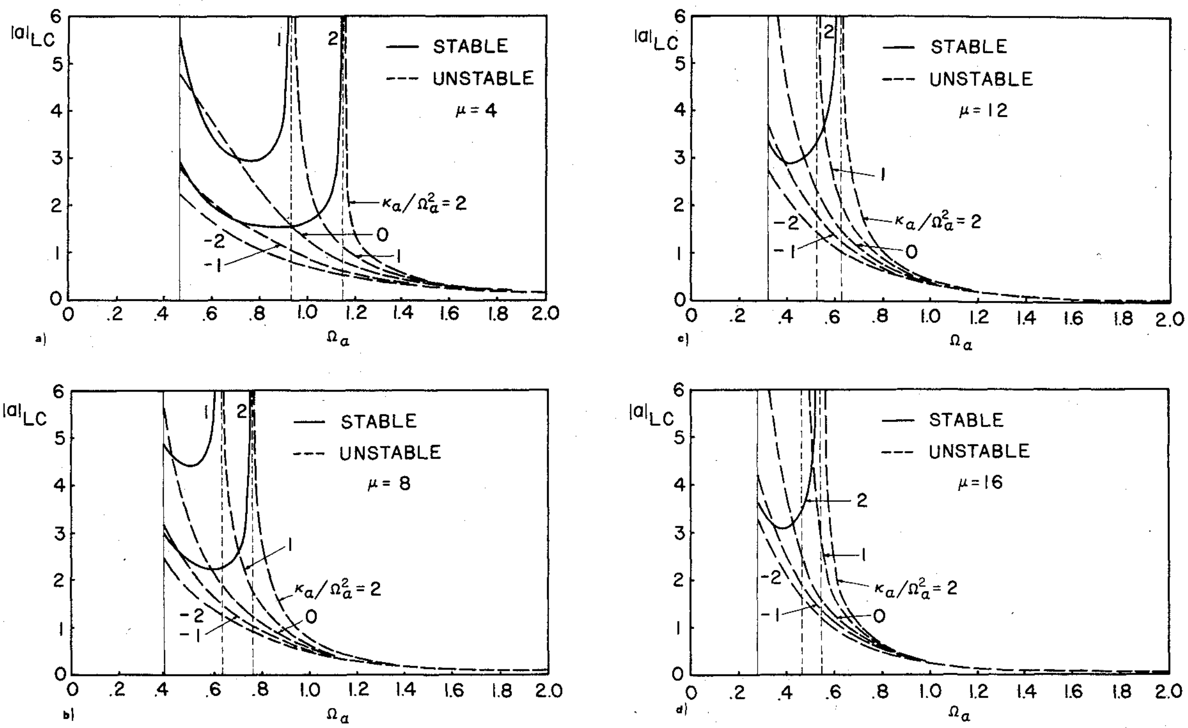

Fig. 12 Limit cycle value of $|a|$, for airfoil flutter equations, as a function of Ω_α.

where

$$f_2=\left\{ \sum_{pq} b_{npq}(\lambda_0)x_{1,p}x_{1,q}\right\} \tag{A6}$$

$$f_3=\left\{ \sum_{pq} b_{npq}(\lambda_0)\,(x_{1,p}x_{2,q}+x_{1,q}x_{2,p}) + \sum_{pqr} c_{npqr}(\lambda_0)x_{1,p}x_{1,q}x_{1,r}\right\} \tag{A7}$$

The solution of Eq. (A3) is given by Eq. (11), where a is now a function of τ_1, τ_2,.... By combining Eqs. (A4) and (11), one obtains

$$\partial x_2/\partial\tau_0 - A_0 x_2 = a^2 f_2^{(2)} e^{i2\omega_0\tau_0} + a\bar{a} f_2^{(0)} + \bar{a}^2 \bar{f}_2^{(2)} e^{-i2\omega_0\tau_0} - (\partial a/\partial\tau_1)\, u e^{i\omega_0\tau_0} - (\partial\bar{a}/\partial\tau_1)\,\bar{u} e^{-i\omega_0\tau_0} \tag{A8}$$

where

$$f_2^{(2)}=\left\{ \sum_{pq} b_{npq}(\lambda_0)u_p u_q \right\} \tag{A9}$$

$$f_2^{(0)}=\left\{ \sum_{pq} b_{npq}(\lambda_0)\,(u_p\bar{u}_q+\bar{u}_p u_q)\right\} \tag{A10}$$

The condition to avoid secular terms is $\partial a/\partial\tau_1=0$, and then the solution for x_2 is given by

$$x_2 = a^2 p_2^{(2)} e^{i2\omega_0\tau_0} + a\bar{a} p_2^{(0)} + \bar{a}^2 \bar{p}_2^{(2)} e^{-i2\omega_0\tau_0} \tag{A11}$$

where

$$p_2^{(2)} = [i2\omega_0 I - A_0]^{-1} f_2^{(2)} \tag{A12}$$

$$p_2^{(0)} = -A_0^{-1} f_2^{(0)} \tag{A13}$$

Combining Eqs.(11,A7 and A11) one obtains

$$f_3 = a^3 f_3^{(3)} e^{i3\omega_0\tau_0} + a^2\bar{a} f_3^{(1)} e^{i\omega_0\tau_0} + a\bar{a}^2 \bar{f}_3^{(1)} e^{-i\omega_0\tau_0} + \bar{a}^3 \bar{f}_3^{(3)} e^{-i3\omega_0\tau_0} \tag{A14}$$

where

$$f_3^{(3)}=\left\{ \sum_{pq} b_{npq}(\lambda_0)\,(u_p p_{2,q}^{(2)} + u_q p_{2,p}^{(2)}) + \sum_{pqr} c_{npqr}(\lambda_0)u_p u_q u_r\right\} \tag{A15}$$

$$f_3^{(1)}=\left\{ \sum_{pq} b_{npq}(\lambda_0)\,(\bar{u}_p p_{2,q}^{(2)} + \bar{u}_q p_{2,p}^{(2)} + u_p p_{2,q}^{(0)} + u_q p_{2,p}^{(0)}) + \sum_{pqr} c_{npqr}(\lambda_0)\,(u_p u_q \bar{u}_r + u_p \bar{u}_q u_r + \bar{u}_p u_q u_r)\right\} \tag{A16}$$

Finally, combining Eqs. (11, A5, All, and A14), one obtains Eq. (13), where $z_3^{(3)}$ and $z_3^{(1)}$ are still given by Eqs. (14) and (15), but $f_3^{(3)}$ and $f_3^{(1)}$ now are given by Eqs. (A15) and (A16) instead of Eqs. (16) and (17). Hence, the conclusions are the same as the ones given in the main body of this paper, except that the solution is given by Eq. (33), with error of order ϵ^2 (instead of ϵ^3). Note that the only effects of the second-order nonlinear terms on Eq. (33) are the order of magnitude of the "error" and the different definition of $f_3^{(1)}$ [Eq. (A16) instead of Eq. (17)] which is used in the definition of γ.

Appendix B

In this Appendix, the nonlinear equations governing the flutter of a simply-supported rectangular panel in a supersonic stream are presented. These equations, similar to those obtained in Ref. 9, are in the form used in Ref. 1, which is convenient for a multiple-time-scaling analysis. Note that the piston theory is used here for the evaluation of the aerodynamic pressure. These equations are obtained by assuming that the motion of the panel is governed by the von Karman large-deflection equations for homogeneous plates, that the material is viscoelastic, that external damping is included, and that the aerodynamic pressure is given by the piston theory and by using the Galerkin method, with N sinusoidal modes in the x-direction (direction of the flow) and only one mode in the other direction:

$$\frac{w}{h} = \sum_{n=1}^{N} w_n \sin\left[\frac{n\pi x}{\ell_x}\right]\sin\left[\frac{\pi y}{\ell_y}\right] \tag{B1}$$

where w is the vertical displacement, h is the thickness of the plate, and ℓ_x and ℓ_y are the plate sizes in the x and y directions. If D is the rigidity of the plate, N_x and N_y are the applied tensions in the x and y directions, $E(1+g_s\partial/\partial\tau)$ is the complex Young modulus for the viscoelastic material, ρ_m is its density, ρ_∞, a_∞, and U_∞ are the density, the speed of sound, and the velocity of the undisturbed air, then the nonlinear panel flutter equations are

$$\ddot{w}+G\dot{w}+(K+\lambda E)w=f \tag{B2}$$

where overdots indicate differentiation with respect to the nondimensional time $\tau=t/T$, with

$$T=(\rho_m h\,\ell_x^4/D)^{1/2} \tag{B3}$$

The matrix G is given by

$$G=[\,g_n\,]=[\,\theta_1+\theta_2(n^2+\ell_x^2/\ell_y^2)^2\,] \tag{B4}$$

with

$$\theta_1=(g_E+\rho_\infty a_\infty)\ell_x^4/DT,\ \theta_2=\pi^4 g_s/T \tag{B5}$$

the matrix K is given by

$$K=[\Omega_n^2]=[n^4(n^2+\ell_x^2/\ell_y^2)^2+\pi^2(R_x n^2+R_y\ell_x^2/\ell_y^2)] \tag{B6}$$

with $R_x=N_x\ell_x^2/D$, $R_y=N_y\ell_x^2/D$, whereas the aerodynamic matrix is given by $E=[e_{np}]$, with

$$e_{np}=4np/(n^2-p^2)\quad \text{if } n+p=\text{odd}$$
$$=0\quad \text{if } n+p=\text{even} \tag{B7}$$

and the parameter λ is given by

$$\lambda=\rho_\infty U_\infty a_\infty \ell_x^3/D \tag{B8}$$

In addition,

$$\tilde{f}=\{\tilde{f}_n\}=\left\{\sum_{pqr}\tilde{c}_{npqr}w_p w_q w_r\right\} \tag{B9}$$

where, indicating with ν_P the Poisson's ratio,

$$\tilde{c}_{npqr}=-\frac{3}{2}\pi^4\left[n^2q^2+\nu_P(n^2+q^2)\frac{\ell_x^2}{\ell_y^2}+\frac{\ell_x^4}{\ell_y^4}\right]\delta_{pq}\delta_{nr}$$
$$-\frac{3}{4}\pi^4(1-\nu_P^2)\frac{\ell_x^4}{\ell_y^4}\left\{\frac{4qr(p^2-q^2)}{[(p+q)^2+4\ell_x^2/\ell_y^2]^2}(\beta_{p+q}^{n+r}+\beta_{p+q}^{n-r})\right.$$
$$+\frac{4qr(p^2-q^2)}{[(p-q)^2+4\ell_x^2/\ell_y^2]^2}(\beta_{p-q}^{n+r}+\beta_{p-q}^{n-r})$$
$$+\left[\frac{2q}{p+q}-q(p-q)\frac{(p+q)^2+4r^2}{[(p+q)^2+4\ell_x^2/\ell_y^2]^2}\right](\gamma_{p+q}^{n+r}-\gamma_{p+q}^{n-r})$$
$$\left.+\left[\mu_{pq}-q(p+q)\frac{(p-q)^2+4r^2}{[(p-q)^2+4\ell_x^2/\ell_y^2]^2}\right](\gamma_{p-q}^{n+r}-\gamma_{p-q}^{n-r})\right\} \tag{B10}$$

with (indicating with $\delta_{i,h}$ the Kronecker delta)

$$\beta_i^h=\delta_{i,h}-\delta_{i,-h},\ \gamma_i^h=\delta_{i,h}+\delta_{i,-h} \tag{B11}$$

and

$$\mu_{pq}=2q/(p-q)\quad \text{if } p\neq q$$
$$=0\quad \text{if } p=q \tag{B12}$$

It is worth noting that the coefficients depend only upon seven parameters: λ, ℓ_x/ℓ_y, θ_1, θ_2, R_x, R_y, and ν_P. Note also that U_∞ appears only in λ. Therefore, it seems to be justified to replace, in θ_1, the aerodynamic-damping parameter $(\lambda\mu/M)^{1/2}$ with $\rho_\infty a_\infty \ell_x^4/DT\equiv(\lambda\mu/M)^{1/2}$ and treat this as independent of λ. This implies considerable simplifications (compare to Refs. 1 and 4).

Equation (A2) may be rewritten in state variable form, Eq. (1), with

$$x=\left\{\begin{array}{c} w \\ \hline \dot{w}\end{array}\right\} \tag{B13}$$

$$A=-\left[\begin{array}{c|c} 0 & -I \\ \hline K & G\end{array}\right]-\lambda\left[\begin{array}{c|c} 0 & 0 \\ \hline E & 0\end{array}\right] \tag{B14}$$

$$f=\left\{\begin{array}{c} 0 \\ \hline \tilde{f}\end{array}\right\} \tag{B15}$$

Appendix C

In this Appendix, the nonlinear equations, which govern the flutter of an airfoil in a supersonic stream, are presented. A slightly more general case is considered in detail in Ref. 5. Assuming that the center of mass CM moves vertically, the motion is described by the horizontal and vertical displacements u and w:

$$u=(\xi-\xi_c)(\cos\alpha-1),\ w=h-(\xi-\xi_c)\sin\alpha \tag{C1}$$

where h is the vertical displacement of CM, α is the rotation angle around CM, ξ is the coordinate along the airfoil, and ξ_c is the abscissa of CM. The Lagrange equations of motion are

$$m d^2h/dt^2+\partial U/\partial h=Q_h \tag{C2a}$$

$$I d^2\alpha/dt^2+\partial U/\partial\alpha=Q_\alpha \tag{C2b}$$

where m and I are the mass and the moment of inertia of the airfoil, and U is the elastic energy of the system, whereas the generalized forces Q_h and Q_α are given by

$$Q_h=-\oint p n_z ds \tag{C3a}$$

$$Q_\alpha=\oint p\,(\sin\alpha\, n_x+\cos\alpha\, n_z)(\xi-\xi_c)ds \tag{C3b}$$

where p is the pressure, n_x and n_z are the components of the normal to the airfoil, and s is the arclength along the contour of the airfoil. If the stiffness of the system is assumed to be represented by two nonlinear (but symmetric) springs located at $\xi_E=\xi_c-\Delta$, the elastic energy may be expressed as

$$U=\tfrac{1}{2}K_h w_E^2+\tfrac{1}{4}K_h' w_E^4+O(w_E^6)+\tfrac{1}{2}K_\alpha\alpha^2+\tfrac{1}{4}K_\alpha'\alpha^4+O(\alpha^6) \tag{C4}$$

where $w_E=h+\Delta\sin\alpha$ is the vertical displacement at ξ_E, whereas K_h, K_h', K_α and K_α' are constants.

If the nonlinear piston theory is used, the pressure distribution is given by

$$p=p_\infty[1+\tfrac{1}{2}(c_p/c_v-1)W/a_\infty]^{2cp/(cp-cv)} \tag{C5}$$

where the downwash W is given by

$$W=(1/|n_z|)\{[(dh/dt)-(\xi-\xi_c)\cos\alpha\dot{\alpha}]n_z-U_\infty n_x] \tag{C6}$$

a_∞ and U_∞ are the speed of sound and the velocity of the freestream, and c_p and c_v are the specific heats of the air.

Equations (C1-C6) fully describe the problem. Considering a zero-thickness, uncambered airfoil, combining the preceding equations, neglecting higher order terms, and indicating with b the half chord, one obtains

$$\begin{bmatrix} 1 & 0 \\ 0 & \nu \end{bmatrix}\begin{Bmatrix} \ddot{\zeta} \\ \ddot{\alpha} \end{Bmatrix} + 6\begin{bmatrix} S_0 & -S_1 \\ -S_1 & S_2 \end{bmatrix}\begin{Bmatrix} \dot{\zeta} \\ \dot{\alpha} \end{Bmatrix} + \begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix}\begin{Bmatrix} \zeta \\ \alpha \end{Bmatrix} + \lambda\begin{bmatrix} 0 & -S_0 \\ 0 & S_1 \end{bmatrix}\begin{Bmatrix} \zeta \\ \alpha \end{Bmatrix} = \begin{Bmatrix} f_\zeta \\ f_\alpha \end{Bmatrix} + \begin{Bmatrix} g_\zeta \\ g_\alpha \end{Bmatrix} \quad \text{(C7)}$$

where $\zeta = h/b$, whereas the over dots indicate differentiation with respect to $\tau = a_\infty t/b$. Furthermore,

$$\nu = I/mb^2 \quad \text{(C8)}$$

and

$$\sigma = 1/\mu = 4\rho_\infty b^2/m \quad \text{(C9a)}$$

$$\lambda = M/\mu = U_\infty 4\rho_\infty b^2/a_\infty m \quad \text{(C9b)}$$

where ρ_∞ is the freestream density, whereas

$$\begin{bmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{bmatrix} = \begin{bmatrix} \Omega_\zeta^2 & \Omega_\zeta^2\delta \\ \Omega_\zeta^2\delta & \Omega_\zeta^2\delta^2 + \nu\Omega_\alpha^2 \end{bmatrix} \quad \text{(C10)}$$

(with $\delta = \Delta/b$, $\Omega_\zeta^2 = K_h b^2/ma_\infty^2$ $\Omega_\alpha^2 = K_\alpha b^2/Ia_\infty^2$), and

$$S_n = \frac{1}{2b}\int_{-b}^{b}\left[\frac{\xi - \xi_c}{b}\right]^n \mathrm{d}\xi = \frac{1}{2}\int_{-1}^{1}(\eta - \eta_c)^n \mathrm{d}\eta \quad \text{(C11)}$$

which yields $S_0 = 1$, $S_1 = -\eta_c$, and $S_2 = 1/3 + \eta_c^2$, with $\eta_c = \xi_c/b$. In addition,

$$f_\zeta = 1/6\, K_{12}\alpha^3 - \kappa_\zeta(\zeta + \delta\alpha)^3 \quad \text{(C12a)}$$

$$f_\alpha = 1/2 K_{12}\zeta\alpha^2 + 2/3\,\Omega_\zeta^2\delta^2\alpha^3 - \delta\kappa_\zeta(\zeta + \delta\alpha)^3 - \nu\kappa_\alpha\alpha^3 \quad \text{(C12b)}$$

$$g_\zeta = \sigma(-1\times 6\, S_0 M\alpha^3 + 1/2 S_0\alpha^2\dot{\zeta} - S_1\alpha^2\dot{\alpha}) + \chi\sigma[S_0(M\alpha - \dot{\zeta})^3$$
$$+ 3S_1(M\alpha - \tilde{\zeta})^2\dot{\alpha} + 3S_2(M\alpha - \dot{\zeta})\dot{\alpha}^2 + S_3\dot{\alpha}^3] \quad \text{(C12c)}$$

$$g_\alpha = \sigma(-1/3 S_1 M\alpha^3 + 1/2 S_2\alpha^2\dot{\alpha}) - \chi\sigma[S_1(M\alpha - \dot{\zeta})^3$$
$$+ 3S_2(M\alpha - \dot{\zeta})^2\dot{\alpha} + 3S_3(M\alpha - \dot{\zeta})\dot{\alpha}^2 + S_4\dot{\alpha}^3] \quad \text{(C12d)}$$

where $\kappa_\xi = K_h' b^4 ma_\infty^2$ $\kappa_\alpha = K_\alpha' b^2/Ia_\infty^2$ and $\chi = (c_p/c_v + 1)/12$. For the air, $c_p/c_v = 1.4$ and $\chi = 0.2$.

Note that U_∞ appears only in λ. Equation (C7) is equivalent to Eq. (1), with

$$x^T = \lfloor \zeta, \alpha, \dot{\zeta}, \dot{\alpha} \rfloor \quad \text{(C13)}$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -K_{11} & -K_{12} & -\sigma S_0 & \sigma S_1 \\ -K_{12}/\nu & -K_{22}/\nu & \sigma S_1/\nu & -\sigma S_2/\nu \end{bmatrix} + \lambda\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & S_0 & 0 & 0 \\ 0 & -S_1/\nu & 0 & 0 \end{bmatrix} \quad \text{(C14)}$$

and

$$f^T = \lfloor 0, 0, f_\zeta, f_\alpha/\nu \rfloor + \lfloor 0, 0, g_\zeta, g_\alpha/\nu \rfloor \quad \text{(C15)}$$

It may be worth noting that Eq. (C14) yields

$$\lambda_0 = \frac{\nu(\omega_0^2 - \Omega_\zeta^2)(\Omega_\alpha^2 - \omega_0^2) + \delta^2\Omega_\zeta^2\omega_0^2 + \sigma^2\omega_0^4/3}{\omega_0^2\eta_c - \Omega_\zeta^2\eta_E} \quad \text{(C16)}$$

where $\eta_E = \eta_c - \delta$, and

$$\omega_0^2 = \frac{(\eta_E^2 + 1/3)\Omega_\zeta^2 + \nu\Omega_\alpha^2}{\eta_c^2 + 1/3 + \nu} \quad \text{(C17)}$$

As shown in detail in Ref. 5, Eqs. (C16) and (C17) coincide with the results of Ref. 10, for zero-thickness airfoil (i.e., $\bar{A} = \bar{B} = 0$).

Appendix D

In this Appendix, it is shown that the multiple-time-scaling solution obtained in Ref. 1 may be obtained as a particular case of the solution presented here. A slightly more general case is given in detail in Ref. 6, Sec. 3.5. The equations of Ref. 1 are of the type given by Eq. (B 2), which may be rewritten as Eq. (1), with x, A, and f given by Eqs. (B13-B15) respectively. Thus, by setting

$$u = \begin{Bmatrix} \tilde{u} \\ \text{----} \\ \check{u} \end{Bmatrix} \quad \text{(D1)}$$

and combining with Eq. (12), one obtains

$$\check{u} = i\omega_0\tilde{u} \quad \text{(D2)}$$

$$M(\lambda_0)\tilde{u} = [-\omega_0^2 I + i\omega_0 G + K + \lambda_0 E]\tilde{u} = 0 \quad \text{(D3)}$$

Similarly, by setting

$$v^T = [\check{v}^T, \tilde{v}^T] \quad \text{(D4)}$$

and combining with Eq. (16), one obtains

$$v^T = \tilde{v}^T[i\omega_0 I + G] \quad \text{(D5)}$$

$$\tilde{v}^T[-\omega_0^2 I + i\omega_0 G + K + \lambda_0 E] = 0 \quad \text{(D6)}$$

Thus, combining with Eqs. (20) and (21), one obtains

$$\beta = \lambda_2\tilde{v}^T E\tilde{u}/\alpha \quad \text{(D7)}$$

$$\gamma = \tilde{v}^T h_1/\alpha \quad \text{(D8)}$$

where

$$\alpha = v^T u = \tilde{v}^T(2i\omega_0 I + G)\tilde{u} \quad \text{(D9)}$$

$$h_1 = \left\{\sum_{pqr}\tilde{c}_{npqr}(\tilde{u}_p\tilde{u}_q\bar{\tilde{u}}_r + \tilde{u}_p\bar{\tilde{u}}_q\tilde{u}_r + \bar{\tilde{u}}_p\tilde{u}_q\tilde{u}_r)\right\} \quad \text{(D10)}$$

whereas, according to Eqs. (D3) and (D6), $\tilde{u}$ and $\tilde{v}$ are the right and left proper vectors of $M(\lambda_0)$ corresponding to the proper value $i\omega_0$. Equations (D7-D9) are equivalent to Eq. (20) of Ref. 1 for $j = 1$. (Note the difference in notations: c_{npqr}, α, $-s\beta_1$, γ_1, and A of Ref. 1 correspond to $-\tilde{c}_{npqr}$, α, β, γ, and E.)

Appendix E

For the sake of completeness, the relationship between the present theory and the harmonic balance method is discussed in this Appendix. It will be shown that the same steady-state results are obtained if the harmonic-balance equations are solved by an algebraic singular-perturbation method. However, the present method is more powerful, since it gives additional information about the transient response and the stability of the limit cycle without any additional computation.

For simplicity, even-order nonlinear terms are assumed to be equal to zero. Assuming x to be of the type

$$x(t) = e^{i\omega\tau}y + e^{-i\omega\tau}\bar{y} \quad \text{(E1)}$$

and combining Eqs. (1) and (E1) and balancing the first harmonic $e^{i\omega\tau}$, one obtains

$$[i\omega I - A(\lambda)]y = g(y, \lambda) \quad \text{(E2)}$$

where

$$g(y,\lambda) = \left\{ \sum_{pqr} c_{npqr}(\lambda)\,(y_p y_q \bar{y}_r + y_p \bar{y}_q y_r + \bar{y}_p y_q y_r) \right\} \quad \text{(E3)}$$

Equation (E2) is the harmonic-balance equation. This usually is solved numerically. However, in the neighborhood of $\lambda = \lambda_0$, Eq. (E2) may be solved by an algebraic singular-perturbation method. For, by setting

$$y = \epsilon y_1 + \epsilon^3 y_3 + O(\epsilon^5) \quad \text{(E4)}$$

and

$$\omega = \omega_0 + \epsilon^2 \omega_2 + O(\epsilon^4) \quad \text{(E5)}$$

and using Eq. (6), Eq. (E2) yields

$$[i\omega_0 I - A_0] y_1 = 0 \quad \text{(E6)}$$

$$[i\omega_0 I - A_0] y_3 = -i\omega_2 y_1 + \lambda_2 A_2 y_1 + g_3 \quad \text{(E7)}$$

where $g_3 = g(y_1, \lambda_0)$. Equation (E6) yields

$$y_1 = a_0 u \quad \text{(E8)}$$

[see Eq. (12)], where a_0 is a constant. Therefore, $g_3 = a_0^2 \bar{a}_0 f_3^{(1)}$, and Eq. (E7) reduces to

$$[i\omega_0 I - A_0] y_3 = \bar{z}_3^{(1)} = -i\omega_2 a_0 u + \lambda_2 A_2 a_0 u + a_0^2 \bar{a} f_3^{(1)} \quad \text{(E9)}$$

The solution for Eq. (E9) exists only if $v^T z_3^{(1)} = 0$, i.e., [compare to Eq. (19)],

$$i\omega_2 a_0 + \beta a_0 + \gamma a_0^2 \bar{a}_0 = 0 \quad \text{(E10)}$$

where β and γ are given by Eq. (20) and (21). Separating the real part and imaginary part and solving for $|a_0|$ and ω_2, one obtains

$$|a_0| = (-\beta_R/\gamma_R)^{1/2} \quad \text{(E11)}$$

$$\omega_2 = -(\beta_I + \gamma_I |\dot{a}_0|^2) = \beta_R \gamma_I / \gamma_R - \beta_I \quad \text{(E12)}$$

Combining Eq. (E4, E5, E8, E11, and E12) one obtains

$$x = 2\epsilon(-\beta_R \gamma_R)^{1/2} \text{Real}[e^{i(\omega\tau + \varphi_*)} u] + O(\epsilon^3) \quad \text{(E13)}$$

where ω is given by Eq. (34) (with the same expression for ω_2) and φ_* is a constant. Noting that $|\lambda - \lambda_0|^{1/2} = \epsilon + O(\epsilon^3)$, Eq. (E13) coincides with the steady-state solution, Eq. (35). Note that the more general transient solution, Eq. (33), requires the same numerical computation as Eq. (35). Therefore, as mentioned previously, the multiple-time-scaling method yields additional information about transient response and stability of the limit cycle without any additional computation.

Appendix F

The theory presented in the main body of this paper implicitly assumes that, at $\lambda = \lambda_0$, the linear system has only one pair of purely imaginary conjugate roots. The case of more than two roots simultaneously crossing the imaginary axis is of little practical interest and is not considered here. However, the case of a simple root (but not a double root) crossing the imaginary axis at the origin is important and is outlined briefly here. In this case, $\omega_0 = 0$ (nonoscillating system), and Eq. (11) is replaced by

$$x_1 = a(\tau_2, \ldots) u \quad \text{(F1)}$$

where a and u are real. Similarly, Eq. (13) is replaced by

$$\partial x_3 / \partial \tau_2 - A_0 x_3 = z_3 \quad \text{(F2)}$$

with z_3 given by

$$z_3 = -(\partial a / \partial \tau_2)\, u + a \lambda_2 A_2 u + a^3 f_3^{(1)} \quad \text{(F3)}$$

where

$$f_3^{(1)} = \left\{ \sum_{pqr} c_{npqr} u_p u_q u_r \right\} \quad \text{(F4)}$$

The condition for avoiding secular terms still is given by Eq. (19), where β and γ are now real (i.e., $\beta_I = \gamma_I = 0$). The section "Discussion of Solution" applies to the case $\omega_0 = 0$ as well.

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